Enhanced FP2 Examination (Edexcel)

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Solutions

1.
$$
\begin{aligned}
\int e^{-x}\cos x\,dx &= \operatorname{Re}\left\{\int e^{-x}e^{ix}\,dx\right\}\\
&= \operatorname{Re}\left\{\int e^{x(-1+i)}\,dx\right\}\\
&= \operatorname{Re}\left\{\frac{1}{-1+i}e^{x(-1+i)}+c\right\}\\
&= \operatorname{Re}\left\{\frac{1}{-1+i}e^{-x}e^{ix}+c\right\}\\
&= \operatorname{Re}\left\{\frac{1}{2}\left(-1-i\right)e^{-x}e^{ix}+c\right\}\\
&= \operatorname{Re}\left\{\frac{1}{2}\left(-1-i\right)e^{-x}\left(\cos x+i\sin x\right)+c\right\}\\
&= \frac{1}{2}e^{-x}\left(-\cos x+\sin x\right)+c\\
&= \frac{1}{2}e^{-x}\left(\sin x-\cos x\right)+c
\end{aligned}
$$

2. $P(x)=\tan x$ and $Q(x)=\sin x$. Therefore the integrating factor is

$$e^{\int \tan x\,dx}=e^{-\ln(\cos x)}=e^{\ln(\sec x)}=\sec x$$

Multiply both sides by the integrating factor

$$\frac{dy}{dx}\sec x+y\tan x\sec x=\tan x$$

The left hand side is equal to $\frac{d}{dx}\left[y\sec x\right]$, so integrate both sides with respect to x

$$\frac{d}{dx}\left(y\sec x\right)=\tan x$$

$$\Rightarrow y\sec x=\int \tan x\,dx$$

Evaluate the integral

$$y\sec x=\ln(\sec x)+c$$

Therefore the general solution is

$$y=\cos x\ln(\sec x)+c$$

The initial condition is $y(0)=0$, so

$$0=1\ln(1)+C\Rightarrow C=0$$

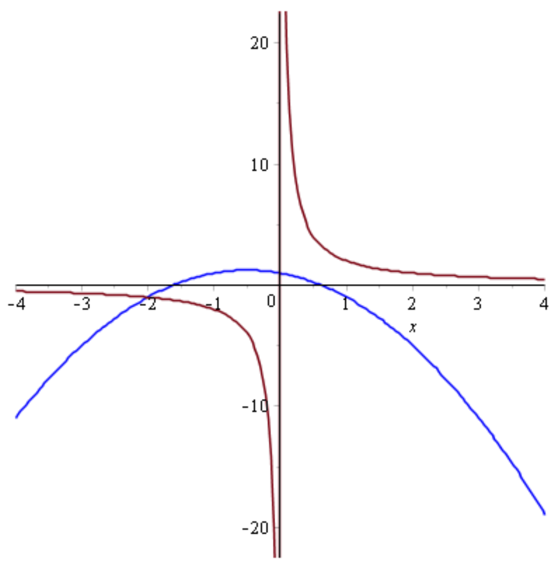
Therefore the solution is

$$y = \cos x \ln(\sec x)$$

3.

$$\begin{aligned} 1 - x - x^2 &> \frac{2}{x} \\ x^2 + x - 1 &< -\frac{2}{x} \\ x^2 + x - 1 + \frac{2}{x} &< 0 \\ \frac{x^3 + x^2 - x + 2}{x} &< 0 \end{aligned}$$

$x = 0$ is a critical value, as are the roots of the cubic polynomial on the numerator. A sketch of the two graphs will help us.



Now there is an asymptote at $x = 0$. So the x values we need are between a small negative value and $x = 0$. The two graphs only meet once, so there is only one real root of $x^3 + x^2 - x + 2$, $x = -2$.

The required x values are $-2 < x < 0$.

4. Let $y = f(x)$. We were given that $y(1) = 0$ and $y'(1) = 1$. Subbing these

into the differential equation yields

$$\frac{d^2y}{dx^2} + \left(0^2 \times 1\right) = 1 \Rightarrow f''(1) = 1$$

Now differentiate both sides of the differential equation implicitly with respect to x.

$$\frac{d^3y}{dx^3} + 2y\left(\frac{dy}{dx}\right)^2 + y^2\frac{dy}{dx} = 1$$

Once again subbing $y(1) = 0$ and $y'(1) = 1$ as well as our newly found $f''(1) = 1$

$$\frac{d^3y}{dx^3} + 0 + 0 = 1 \Rightarrow f'''(1) = 1$$

The formula for a Taylor series is

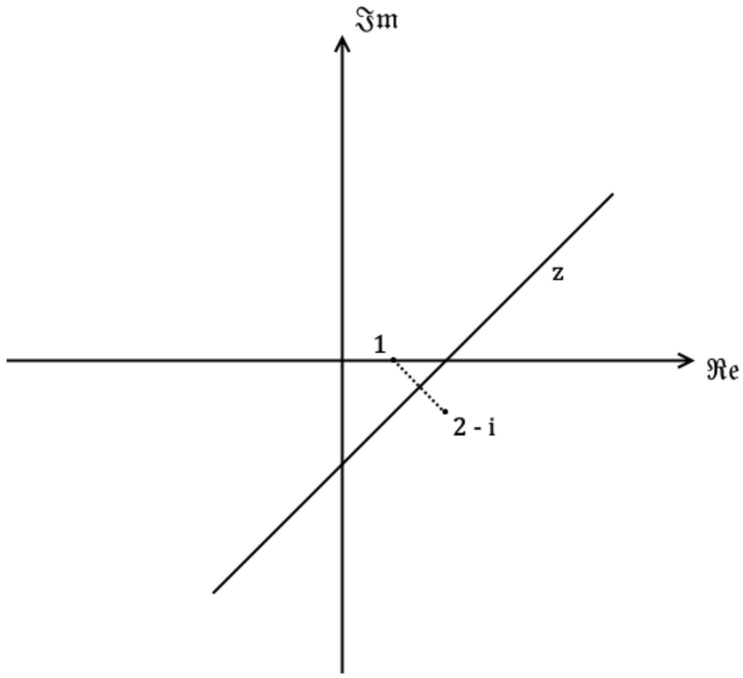
$$f(x) = f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}\,(x-a)^k + \ldots$$

Therefore the required Taylor series for the differential equation is

$$y \approx (x-1) + \frac{1}{2!}\,(x-1)^2 + \frac{1}{3!}\,(x-1)^3$$

5. Let $z = x + yi$

$$\begin{aligned}
|x + yi - 1| &= |x + yi - (2 - i)| \\
|(x-1) + yi| &= |(x-2) + i(y+1)| \\
\Rightarrow \sqrt{(x-1)^2 + y^2} &= \sqrt{(x-2)^2 + (y+1)^2} \\
\Rightarrow (x-1)^2 + y^2 &= (x-2)^2 + (y+1)^2 \\
x^2 - 2x + 1 + y^2 &= x^2 - 4x + 4 + y^2 + 2y + 1 \\
-2x &= -4x + 4 + 2y \\
\therefore y &= x - 2
\end{aligned}$$



6. Start off by finding the complementary function. Letting $y = e^{\lambda x}$ the auxiliary quadratic equation is

$$\lambda^2 - 4\lambda + 4 = 0$$

$$\Rightarrow (\lambda - 2)^2$$

The auxiliary quadratic equation has a repeated real root $\lambda = 2$. Therefore the complementary function is

$$y = (A + Bx)\, e^{2x}$$

Where A and B are arbitrary constants. Now find the particular integral. $f(x) = \sin x$ so our trial function is $y = a\cos x + b\sin x$. Then

$$\frac{dy}{dx} = -a\sin x + b\cos x \quad \text{and} \quad \frac{d^2y}{dx^2} = -a\cos x - b\sin x$$

Subbing these back into the original differential equation

$$(-a\cos x - b\sin x) - 4\,(-a\sin x + b\cos x) + 4\,(a\cos x + b\sin x)$$

$$= (-a - 4b + 4a)\cos x + (-b + 4a + 4b)\sin x$$
$$= (3a - 4b)\cos x + (4a + 3b)\sin x$$

Compare this with the right hand side of the differential equation $\sin x$. Then we get two simultaneous equations

$$3a - 4b = 0$$

$$4a + 3b = 1$$

Solving them yields $a = \frac{4}{25}$ and $b = \frac{3}{25}$. Therefore the general solution is

$$y = (A + Bx)\, e^{2x} + \frac{4}{25}\cos x + \frac{3}{25}\sin x$$

But the boundary conditions state that $y(0) = 0$ and $y'(0) = 0$. Using these boundary conditions yields two simultaneous equations

$$(A + 0)\, e^{0} + \frac{4}{25}\cos(0) + \frac{3}{25}\sin(0) \Rightarrow A + \frac{4}{25} = 0 \Rightarrow A = -\frac{4}{25}$$

Find y' using the product rule

$$y' = Be^{2x} + 2e^{2x}\,(A + Bx) - \frac{4}{25}\sin x + \frac{3}{25}\cos x$$
$$= e^{2x}\,(B + 2A + 2Bx) - \frac{4}{25}\sin x + \frac{3}{25}\cos x$$
$$= e^{2x}\,[2A + B\,(1 + 2x)] - \frac{4}{25}\sin x + \frac{3}{25}\cos x$$

Now we can apply the boundary condition $y'(0) = 0$

$$e^{0}\,[2A + B\,(1 + 0)] - \frac{4}{25}\sin(0) + \frac{3}{25}\cos(0) = 0$$

$$\Rightarrow 2\left(-\frac{4}{25}\right) + B + \frac{3}{25} = 0 \Rightarrow B = \frac{1}{5}$$

Therefore the particular solution is

$$y = \left(\frac{1}{5}x - \frac{4}{25}\right) e^{2x} + \frac{4}{25}\cos x + \frac{3}{25}\sin x$$

7. a. Points in polar coordinates that have horizontal tangent lines satisfy

$$\frac{d}{d\theta}(r\sin\theta) = 0$$

Sub r in

$$\frac{d}{d\theta}\left(e^{\theta}\sin\theta\sin\theta\right) = \frac{d}{d\theta}\left(e^{\theta}\sin^2\theta\right)$$

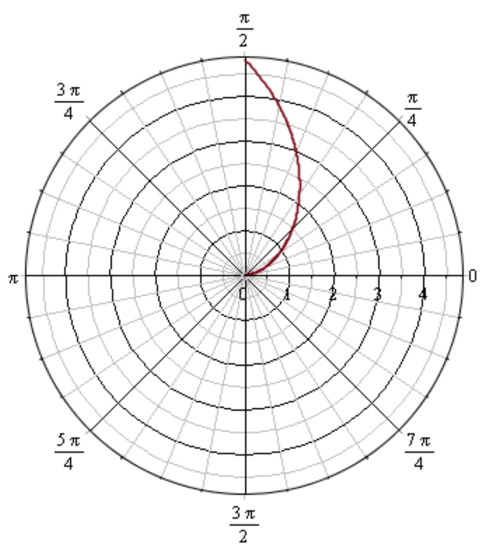
Using the product rule

$$\begin{aligned}\frac{d}{d\theta}\left(e^{\theta}\sin^2\theta\right) &= 2e^{\theta}\cos\theta\sin\theta + e^{\theta}\sin^2\theta \\ &= e^{\theta}\sin\theta\,(2\cos\theta + \sin\theta)\end{aligned}$$

$e^{\theta} \neq 0$ so $\sin\theta = 0 \Rightarrow \theta = k\pi, k \in \mathbb{Z}$.

Also $2\cos\theta + \sin\theta = 0 \Rightarrow \tan\theta = -2$. Pick θ that satisfy this, $\tan^{-1}(-2) = -1.107\ldots$, and $\tan^{-1}(-2) + k\pi, k \in \mathbb{Z}$ work.

b. Sketch points for θ from 0 to $\pi/2$ and you should get something like this:



8. Using partial fractions $\frac{2}{(r-3)(r-1)} = \frac{1}{r-3} - \frac{1}{r-1}$. Compare numerator coefficients or use the Heaviside cover-up method to do this.

Then counting from $r = 5, \ldots, 25$ the sum telescopes so you can match up terms and eliminate them

$$
\begin{aligned}
\frac{1}{r-3} - \frac{1}{r-1} &= \left[\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{20} + \frac{1}{21} + \frac{1}{22}\right] \\
&\quad - \left[\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots + \frac{1}{22} + \frac{1}{23} + \frac{1}{24}\right] \\
&= \frac{1}{2} + \frac{1}{3} - \frac{1}{23} - \frac{1}{24} \\
&= \frac{413}{552}
\end{aligned}
$$