Enhanced FP2 Examination (MEI)

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Solutions

1.

$$\begin{aligned}
\mathbf{A}^n &= \underbrace{\mathbf{AAA}\cdots\mathbf{A}}_{\text{n times}} \\
&= \underbrace{\left(\mathbf{PDP}^{-1}\right)\left(\mathbf{PDP}^{-1}\right)\left(\mathbf{PDP}^{-1}\right)\cdots\left(\mathbf{PDP}^{-1}\right)}_{\text{n times}} \\
&= \mathbf{PD}\left(\mathbf{P}^{-1}\mathbf{P}\right)\mathbf{D}\left(\mathbf{P}^{-1}\mathbf{P}\right)\cdots\mathbf{D}\left(\mathbf{P}^{-1}\mathbf{P}\right)\mathbf{DP}^{-1} \\
&= \mathbf{PDIDI}\cdots\mathbf{DIDP}^{-1} \\
&= \mathbf{P}\underbrace{\mathbf{DDDD}}_{\text{n times}}\mathbf{P}^{-1} \\
&= \mathbf{PD}^n\mathbf{P}^{-1}
\end{aligned}$$

2. First off we have the identity $1 - \tanh^2 x \equiv \operatorname{sech}^2 x$ so

$$\operatorname{sech}^2(2x) = 1 - \tanh^2(2x)$$

By Osborn's rule we can find the identity

$$\tanh(A+B) = \frac{\tanh A + \tanh B}{1 + \tanh A \tanh B}$$

Therefore

$$\tanh(2x) = \tanh(x+x) = \frac{\tanh x + \tanh x}{1 + \tanh x \tanh x} = \frac{2\tanh x}{1 + \tanh^2 x}$$

Going back to the sech identity

$$\operatorname{sech}^2(2x) = 1 - \tanh^2(2x) = 1 - \left(\frac{2\tanh x}{1 + \tanh^2 x}\right)^2$$

Making a common denominator

$$1 - \frac{4\tanh^2 x}{\left(1 + \tanh^2 x\right)^2} = \frac{\left(1 + \tanh^2 x\right)^2 - 4\tanh^2 x}{\left(1 + \tanh^2 x\right)^2}$$

Expanding the square term in the numerator

$$\frac{\left(1 + \tanh^2 x\right)^2 - 4\tanh^2 x}{\left(1 + \tanh^2 x\right)^2} = \frac{1 + 2\tanh^2 x + \tanh^4 x - 4\tanh^2 x}{\left(1 + \tanh^2 x\right)^2}$$

Then finally simplifying the numerator and factorising the numerator

$$\frac{\tanh^4 x - 2\tanh^2 x + 1}{\left(1 + \tanh^2 x\right)^2} = \frac{\left(\tanh^2 x - 1\right)^2}{\left(\tanh^2 x + 1\right)^2} = \left(\frac{\tanh^2 x - 1}{\tanh^2 x + 1}\right)^2$$

3. Complete the square in the denominator

$$8 - 2x - x^2 = 8 - \left(2x + x^2\right) = 8 - \left((x+1)^2 - 1\right) = 9 - (x+1)^2$$

Then the integral is in the standard form. Let $u = x + 1$ so that $\frac{du}{dx} = 1$

$$\int \frac{1}{\sqrt{9-(x+1)^2}}\, dx = \int \frac{1}{\sqrt{9-u^2}}\, du = \arcsin\left(\frac{u}{3}\right) + c = \arcsin\left(\frac{x+1}{3}\right) + c$$

where c is a constant of integration.

4. a. Let $z = 3 - i$. Then $|z| = \sqrt{3^2+1} = \sqrt{10}$ and $\arg z = -\tan^{-1}\frac{1}{3} \approx -0.322$ to 3 significant figures.

Therefore $z = \sqrt{10}e^{\left(-i\tan^{-1}\frac{1}{3}\right)}$ and the cube roots are $z^{\frac{1}{3}} = 10^{\frac{1}{6}}e^{\left(-i\tan^{-1}\frac{1}{3} + \frac{2k\pi}{3}\right)}$ for $k = 0, \ldots, 2$.

b. Several ways to argue this.

Geometrically: The n roots of order n are equally spaced around a circle. If their sum was nonzero, it would have an argument (an angle relative to the real axis) which would be a violation of symmetry. Therefore, by symmetry, the sum must be 0.

Stretching FP1 a bit: Let the n roots of order n be x_i. By definition they are all solutions of the polynomial $x_i^n - z = 0$. Recall from FP1 that the sum of the roots of a polynomial is equal to -1 times the coefficient of the x_i^{n-1} term, which is 0.

Directly: Let z be a complex number. Then if its modulus is $|z|$ and its argument is θ then its nth roots are

$$z^{\frac{1}{n}} = |z|e^{i\left(\theta + \frac{2k\pi}{n}\right)}$$

there are n of these nth roots and their sum is

$$\sum_{k=0}^{n-1} |z|e^{i\left(\theta + \frac{2k\pi}{n}\right)} = |z|e^{i\theta}\sum_{k=0}^{n-1} e^{i\frac{2k\pi}{n}}$$

Then

$$\sum_{k=0}^{n-1} e^{i\frac{2k\pi}{n}}$$

is a finite geometric series.

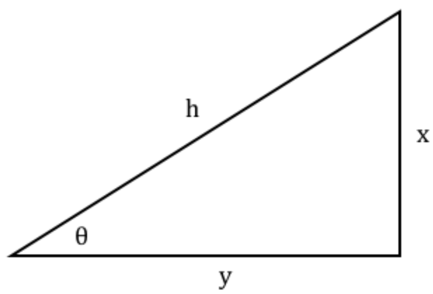
We can express each term in this series as ω^k. Notice that these are the nth roots of unity, i.e. $\omega^n = 1$. They form a sequence $1, \omega, \omega^2, \ldots, \omega^{n-1}$.

As said above this is a finite geometric series, therefore

$$\sum_{k=0}^{n-1} \omega^k = \frac{1-\omega^n}{1-\omega} = 0$$

since there are n of them and $\omega \neq 1$.

5. Let $\theta = \tan^{-1} \frac{x}{y}$, so $\tan\theta = \frac{x}{y}$. Then we can form a right angle triangle with θ as an angle, x as the opposite side's length, and y as the adjacent side's length. Label the hypotenuse as h.



By Pythagoras the unknown hypotenuse is $\sqrt{x^2+y^2}$. Therefore

$$\sin\theta = \frac{x}{\sqrt{x^2+y^2}}$$

as required.

6. Using partial fractions $\frac{2}{(r-3)(r-1)} = \frac{1}{r-3} - \frac{1}{r-1}$. Compare numerator coefficients or use the Heaviside cover-up method to do this.

Then counting from $r = 5, \ldots, 25$ the sum telescopes so you can match up terms and eliminate them

$$\begin{aligned}
\frac{1}{r-3} - \frac{1}{r-1} &= \left[\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{20} + \frac{1}{21} + \frac{1}{22}\right] \\
&\quad - \left[\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots + \frac{1}{22} + \frac{1}{23} + \frac{1}{24}\right] \\
&= \frac{1}{2} + \frac{1}{3} - \frac{1}{23} - \frac{1}{24} \\
&= \frac{413}{552}
\end{aligned}$$

7. Lines invariant under a matrix are scalar multiples of the matrix's eigenvectors. Any point on such lines pre-multiplied by the matrix will stay on that line.

The characteristic polynomial of the matrix is

$$(-1-\lambda)(6-\lambda)+12=\lambda^2-5\lambda+6$$

which can be set to equal zero and factorised into

$$(\lambda-2)(\lambda-3)=0$$

Therefore the eigenvalues are $2, 3$.

For $\lambda=2$:

$$\begin{pmatrix} -3 & 2 \\ -6 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Therefore one line invariant under the matrix is $-3x+2y=0 \Rightarrow y=\frac{3}{2}x$.

For $\lambda=3$:

$$\begin{pmatrix} -4 & 2 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Therefore the second line is $-4x+2y=0 \Rightarrow y=2x$.

8. Suppose that a function $f(x)$ can be expressed as the polynomial

$$f(x)=a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_kx^k+\ldots$$

where a_i are constant coefficients. Then $f(nx)$ can be expressed as the polynomial

$$f(nx)=a_0+a_1nx+a_2(nx)^2+a_3(nx)^3+\ldots+a_k(nx)^k+\ldots$$

where a_i are constant coefficients. Then

$$\begin{aligned} f'(nx) &= a_1n+2a_2n^2x+3a_3n^3x^2+\ldots \\ f''(nx) &= 2a_2n^2+6a_3n^3x+\ldots \\ f'''(nx) &= 6a_3n^3+\ldots \end{aligned}$$

And so on. Then

$$\begin{aligned} f(0) &= a_0 \\ f'(0) &= a_1 n \Rightarrow a_1 = \frac{f'(0)}{n} \\ f''(0) &= 2a_2 n^2 \Rightarrow a_2 = \frac{f''(0)}{2n^2} = \frac{f''(0)}{2!n^2} \\ f'''(0) &= 6a_3 n^3 \Rightarrow a_3 = \frac{f'''(0)}{6n^3} = \frac{f'''(0)}{3!n^3} \end{aligned}$$

And so on. Then we can express $f(nx)$ as

$$f(nx) = f(0) + f'(0)\frac{x}{n} + \frac{f''(0)}{2}\left(\frac{x}{n}\right)^2 + \ldots + \frac{f^{(k)}(0)}{k!}\left(\frac{x}{n}\right)^k + \ldots$$