Further Pure Mathematics 1 (FP1)

Enhanced Practice Examination

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Time Allowed: 1 hour 30 minutes | **Total Marks:** 75 marks

Answer all questions. Show all necessary working; full marks may not be awarded without working.

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1. [Matrices and Proof by Induction] [7 marks]
Given that matrix $\mathbf{M} = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$, prove by mathematical induction that for all positive integers $n \geq 1$:

$$\mathbf{M}^n = \begin{pmatrix} (-1)^n & 0 \\ 1-(-1)^n & 1 \end{pmatrix}$$

2. [Complex Numbers: Roots and Cartesian Form] [7 marks]

(a) Solve the equation $w^2 = 3 - 4i$, giving your solutions in the form $a + bi$ where $a, b \in \mathbb{R}$. [4]

(b) Hence express $\sqrt{3-4i} + \sqrt{8+6i}$ in the form $x + iy$ where $x, y \in \mathbb{R}$, taking the principal square roots with positive real parts. [3]

3. [Summation of Series] [7 marks]

(a) Show that $(3-r)^2 - (r+1)^2 = 8(1-r)$. [2]

(b) Evaluate the summation:

$$S_n = \sum_{r=0}^{n} \left[(3-r)^2 - (r+1)^2\right]$$

giving your answer in fully factorised form in terms of n. [5]

4. [Complex Numbers: Loci in the Argand Plane] [7 marks]
A complex variable $z = x + iy$ satisfies the equation $|z-1| = |z-2+i|$.

(a) Show that the locus of z is a straight line, and determine its Cartesian equation in the form $y = mx + c$. [4]

(b) Find the complex number z_0 on this locus that has the minimum modulus $|z_0|$. [3]

5. [Roots of Polynomials] [9 marks]
The polynomial equation

$$z^4 - 4z^3 + 14z^2 - 4z + 13 = 0$$

has real coefficients and has $z = i$ as one of its roots.

(a) State another complex root of the equation. [1]

(b) Factorise the polynomial completely into two real quadratic factors. [4]

(c) Find the remaining two roots of the equation in exact Cartesian form. [4]

6. [Matrix Algebra and Inverses] [8 marks]
Let $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$.

(a) Show that $\mathbf{A}^2 - 3\mathbf{A} + 2\mathbf{I} = \mathbf{0}$, where $\mathbf{I}$ is the 2×2 identity matrix. [3]

(b) Hence express $\mathbf{A}^{-1}$ as a linear combination of $\mathbf{A}$ and $\mathbf{I}$. [2]

(c) Use this result to compute $\mathbf{A}^{-1}$, and verify your answer using the standard determinant-adjoint formula. [3]

7. [Linear Transformations in 2D] [8 marks]
A linear transformation T in the plane is represented by the matrix:

$$\mathbf{P} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

(a) Describe fully the geometric transformation represented by $\mathbf{P}$. [3]

(b) A second transformation S is represented by $\mathbf{Q} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. The composite transformation R consists of T followed by S. Write down the matrix $\mathbf{R}$ representing R. [2]

(c) Find the image of the straight line $y = 2x + 1$ under the composite transformation R. [3]

8. [Numerical Methods: Linear Interpolation and Newton-Raphson] [10 marks]
Let $f(x) = x^{3/4} + 2x^{1/2} - 1$ for $x \geq 0$.

(a) Show that $f(x) = 0$ has a root α in the interval $[0.1, 0.2]$. [2]

(b) Using linear interpolation between $x = 0.1$ and $x = 0.2$, derive the approximation formula:

$$\alpha \approx \frac{0.1|f(0.2)| + 0.2|f(0.1)|}{|f(0.1)| + |f(0.2)|}$$

and evaluate α correct to 3 decimal places. [4]

(c) Explain why the Newton-Raphson method cannot be initiated with $x_0 = 0$. [2]

(d) Show that $f'(x) > 0$ for all $x > 0$, and deduce that α is the unique root of $f(x) = 0$ on $[0, \infty)$. [2]

9. [Coordinate Geometry: Parabola and Rectangular Hyperbola] [12 marks]
The curve C is a parabola with equation $y^2 = 4ax$, where $a > 0$ is a constant. The point $P(ap^2, 2ap)$ lies on C with $p \neq 0$.

(a) Show that the equation of the tangent to C at P is $py = x + ap^2$. [4]

(b) The tangent at P intersects the rectangular hyperbola H with equation $xy = c^2$ at two distinct points A and B. Find the midpoint of AB in terms of a, c, and p. [4]

(c) Show that as p varies, the locus of the midpoint of AB satisfies $y(2x - y) =$ constant if $a = c$. [4]

END OF EXAMINATION