Further Pure Mathematics 1 (FP1)

Complete Model Solutions & Commentary

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Module: Further Pure Mathematics 1 (FP1) | **Max Mark:** 75 marks

Question 1. [Matrices and Proof by Induction]

Proposition: $P(n) : \mathbf{M}^n = \begin{pmatrix} (-1)^n & 0 \\ 1-(-1)^n & 1 \end{pmatrix}$ for all integers $n \geq 1$, where $\mathbf{M} = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$.

Base Case ($n = 1$):

$$\mathbf{M}^1 = \begin{pmatrix} (-1)^1 & 0 \\ 1-(-1)^1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1-(-1) & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} = \mathbf{M}$$

Hence $P(1)$ is true.

Inductive Hypothesis: Assume $P(k)$ is true for some integer $k \geq 1$, i.e.

$$\mathbf{M}^k = \begin{pmatrix} (-1)^k & 0 \\ 1-(-1)^k & 1 \end{pmatrix}$$

Inductive Step: Consider $n = k + 1$:

$$\mathbf{M}^{k+1} = \mathbf{M}^k\mathbf{M} = \begin{pmatrix} (-1)^k & 0 \\ 1-(-1)^k & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$$

Performing matrix multiplication row by column:

- Top-left: $(-1)^k(-1) + 0(2) = (-1)^{k+1}$
- Top-right: $(-1)^k(0) + 0(1) = 0$
- Bottom-left: $(1-(-1)^k)(-1)+1(2) = -1+(-1)^k+2 = 1+(-1)^k = 1-(-1)^{k+1}$
- Bottom-right: $(1-(-1)^k)(0) + 1(1) = 1$

Combining the entries:

$$\mathbf{M}^{k+1} = \begin{pmatrix} (-1)^{k+1} & 0 \\ 1-(-1)^{k+1} & 1 \end{pmatrix}$$

This is of the form $P(k + 1)$.

Conclusion: Since $P(1)$ is true, and $P(k) \implies P(k+1)$, the result holds for all integers $n \geq 1$ by the principle of mathematical induction. ■

Question 2. [Complex Numbers: Roots and Cartesian Form]

(a) Let $w = a + bi$ where $a, b \in \mathbb{R}$.

$$w^2 = (a+bi)^2 = a^2 - b^2 + 2abi = 3 - 4i$$

Equating real and imaginary parts:

$$a^2 - b^2 = 3 \quad \text{and} \quad 2ab = -4 \implies ab = -2 \implies b = -\frac{2}{a}$$

Substitute into the real part:

$$a^2 - \left(-\frac{2}{a}\right)^2 = 3 \implies a^2 - \frac{4}{a^2} = 3 \implies a^4 - 3a^2 - 4 = 0$$

Factoring as a quadratic in a^2:

$$(a^2 - 4)(a^2 + 1) = 0$$

Since $a \in \mathbb{R}$, $a^2 \geq 0$, so $a^2 = 4 \implies a = \pm 2$. When $a = 2$, $b = -2/2 = -1 \implies w = 2 - i$. When $a = -2$, $b = -2/(-2) = 1 \implies w = -2 + i$. Therefore, $w = \pm(2 - i)$.

(b) For $\sqrt{8+6i}$, let $(c+di)^2 = 8 + 6i$.

$$c^2 - d^2 = 8 \quad \text{and} \quad 2cd = 6 \implies cd = 3 \implies d = \frac{3}{c}$$

$$c^2 - \frac{9}{c^2} = 8 \implies c^4 - 8c^2 - 9 = 0 \implies (c^2 - 9)(c^2 + 1) = 0 \implies c = \pm 3, \quad d = \pm 1$$

With positive real parts (principal branches): $\sqrt{3-4i} = 2 - i$ and $\sqrt{8+6i} = 3 + i$. Summing the two roots:

$$\sqrt{3-4i} + \sqrt{8+6i} = (2-i) + (3+i) = 5$$

Final Answer: $5 + 0i = 5$.

Question 3. [Summation of Series]

(a) Expanding the difference of two squares or expanding each bracket directly:

$$\begin{aligned}(3-r)^2-(r+1)^2 &= (9-6r+r^2)-(r^2+2r+1)\\ &= 9-6r+r^2-r^2-2r-1\\ &= 8-8r = 8(1-r) \quad \text{as required.}\end{aligned}$$

(b) The summation begins at $r = 0$:

$$S_n = \sum_{r=0}^{n} 8(1-r) = 8\sum_{r=0}^{n}(1-r)$$

Separate the $r = 0$ term, or use standard summation rules over $n+1$ terms:

$$\sum_{r=0}^{n} 1 = n+1 \quad \text{and} \quad \sum_{r=0}^{n} r = \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

Substituting:

$$\begin{aligned}S_n &= 8\left[(n+1)-\frac{n(n+1)}{2}\right]\\ &= 8(n+1)\left(1-\frac{n}{2}\right)\\ &= 8(n+1)\left(\frac{2-n}{2}\right)\\ &= 4(n+1)(2-n)\end{aligned}$$

Final Answer: $4(n+1)(2-n)$ or $8+4n-4n^2$.

Question 4. [Complex Numbers: Loci in the Argand Plane]

(a) Let $z = x+iy$ with $x, y \in \mathbb{R}$.

$$|x+iy-1| = |x+iy-(2-i)| \implies |(x-1)+iy| = |(x-2)+i(y+1)|$$

Squaring both sides (distances are non-negative):

$$\begin{aligned}(x-1)^2+y^2 &= (x-2)^2+(y+1)^2\\ x^2-2x+1+y^2 &= x^2-4x+4+y^2+2y+1\\ -2x+1 &= -4x+2y+5\\ 2y &= 2x-4\\ y &= x-2\end{aligned}$$

This is a straight line (the perpendicular bisector of the line segment joining $1+0i$ and $2-i$).

(b) The point of minimum modulus on the line $y = x-2$ is the point closest to the origin $(0, 0)$. The line through the origin perpendicular to $y = x-2$ (gradient $m = 1$) has gradient $m_\perp = -1$, giving equation $y = -x$. Intersecting the two lines:

$$-x = x-2 \implies 2x = 2 \implies x = 1, \quad y = -1$$

Final Answer: $z_0 = 1 - i$, with minimum modulus $|z_0| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$.

Question 5. [Roots of Polynomials]

(a) Since the coefficients of $z^4 - 4z^3 + 14z^2 - 4z + 13 = 0$ are real, non-real roots must occur in complex conjugate pairs. Given that $z = i$ is a root, its complex conjugate $z = -i$ is also a root.

(b) The quadratic factor corresponding to roots $\pm i$ is:

$$(z - i)(z + i) = z^2 + 1$$

Dividing the quartic by $z^2 + 1$ using algebraic long division or equating coefficients:

$$z^4 - 4z^3 + 14z^2 - 4z + 13 = (z^2 + 1)(z^2 + pz + q)$$

Comparing coefficients:

- z^3 term: $p = -4$
- Constant term: $1 \times q = 13 \implies q = 13$
- Check z^2 term: $q + 1 = 13 + 1 = 14$ (consistent!)
- Check z term: $p = -4$ (consistent!)

Complete factorisation: $(z^2 + 1)(z^2 - 4z + 13) = 0$.

(c) Solving $z^2 - 4z + 13 = 0$ using the quadratic formula:

$$z = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(13)}}{2} = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

Final Answer: The remaining roots are $z = -i$, $z = 2 + 3i$, and $z = 2 - 3i$.

Question 6. [Matrix Algebra and Inverses]

(a) Compute $\mathbf{A}^2$:

$$\mathbf{A}^2 = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 3(3)+(-1)(2) & 3(-1)+(-1)(0) \\ 2(3)+0(2) & 2(-1)+0(0) \end{pmatrix} = \begin{pmatrix} 7 & -3 \\ 6 & -2 \end{pmatrix}$$

Now evaluate $\mathbf{A}^2 - 3\mathbf{A} + 2\mathbf{I}$:

$$\begin{pmatrix} 7 & -3 \\ 6 & -2 \end{pmatrix} - 3\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 7-9+2 & -3+3+0 \\ 6-6+0 & -2-0+2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$$

(b) From $\mathbf{A}^2 - 3\mathbf{A} + 2\mathbf{I} = \mathbf{0}$, isolate $\mathbf{I}$:

$$2\mathbf{I} = 3\mathbf{A} - \mathbf{A}^2 = \mathbf{A}(3\mathbf{I} - \mathbf{A})$$

Pre-multiplying by $\mathbf{A}^{-1}$ and dividing by 2:

$$\mathbf{A}^{-1} = \frac{1}{2}(3\mathbf{I} - \mathbf{A}) = \frac{3}{2}\mathbf{I} - \frac{1}{2}\mathbf{A}$$

(c) Calculate using the relation:

$$\mathbf{A}^{-1} = \frac{1}{2}\left[\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{pmatrix}$$

Verification by determinant formula: $\det(\mathbf{A}) = (3)(0) - (-1)(2) = 2 \neq 0$.

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{pmatrix} 0 & -(-1) \\ -2 & 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{pmatrix} \quad \checkmark$$

Question 7. [Linear Transformations in 2D]

(a) The matrix is:

$$\mathbf{P} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix}$$

This represents an **anticlockwise rotation of** 45° **(or** $\frac{\pi}{4}$ **radians) about the origin** $(0, 0)$.

(b) The transformation $R = S \circ T$ is represented by $\mathbf{R} = \mathbf{QP}$:

$$\mathbf{R} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$$

(c) Let $\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{R}\begin{pmatrix} x \\ y \end{pmatrix}$. To find the Cartesian equation of the image, express $\begin{pmatrix} x \\ y \end{pmatrix}$ in terms of $\begin{pmatrix} x' \\ y' \end{pmatrix}$ using $\mathbf{R}^{-1}$:

$$\det(\mathbf{R}) = (\sqrt{2})(\sqrt{2}) - (-\sqrt{2})(\sqrt{2}) = 2 + 2 = 4$$

$$\mathbf{R}^{-1} = \frac{1}{4}\begin{pmatrix} \sqrt{2} & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{pmatrix} \implies \begin{pmatrix} x \\ y \end{pmatrix} = \frac{\sqrt{2}}{4}\begin{pmatrix} x' + y' \\ -x' + y' \end{pmatrix}$$

Substitute $x = \frac{\sqrt{2}}{4}(x' + y')$ and $y = \frac{\sqrt{2}}{4}(-x' + y')$ into $y = 2x + 1$:

$$\frac{\sqrt{2}}{4}(-x' + y') = 2\left[\frac{\sqrt{2}}{4}(x' + y')\right] + 1$$

Multiply through by $\frac{4}{\sqrt{2}} = 2\sqrt{2}$:

$$-x' + y' = 2x' + 2y' + 2\sqrt{2} \implies y' = -3x' - 2\sqrt{2}$$

Final Answer: The image is the straight line $y = -3x - 2\sqrt{2}$.

Question 8. [Numerical Methods]

(a) $f(x) = x^{3/4} + 2x^{1/2} - 1$. Evaluate at interval endpoints:

$$f(0.1) = (0.1)^{0.75} + 2(0.1)^{0.5} - 1 \approx 0.1778 + 2(0.3162) - 1 = -0.1897 < 0$$
$$f(0.2) = (0.2)^{0.75} + 2(0.2)^{0.5} - 1 \approx 0.2991 + 2(0.4472) - 1 = +0.1935 > 0$$

Since $f(x)$ is continuous on $[0.1, 0.2]$ and changes sign, by the Intermediate Value Theorem there exists at least one root $\alpha \in (0.1, 0.2)$.

(b) Under linear interpolation, the curve between $(0.1, f(0.1))$ and $(0.2, f(0.2))$ is approximated by a chord. Using similar right-angled triangles with the x-axis: The lower triangle has base $\alpha - 0.1$ and height $|f(0.1)|$. The upper triangle has base $0.2 - \alpha$ and height $|f(0.2)| = f(0.2)$.

$$\frac{\alpha - 0.1}{|f(0.1)|} = \frac{0.2 - \alpha}{f(0.2)} \implies f(0.2)(\alpha - 0.1) = |f(0.1)|(0.2 - \alpha)$$

$$\alpha \left(f(0.2) + |f(0.1)|\right) = 0.1f(0.2) + 0.2|f(0.1)| \implies \alpha \approx \frac{0.1f(0.2) + 0.2|f(0.1)|}{f(0.2) + |f(0.1)|}$$

Substituting values:

$$\alpha \approx \frac{0.1(0.1935) + 0.2(0.1897)}{0.1935 + 0.1897} = \frac{0.01935 + 0.03794}{0.3832} = \frac{0.05729}{0.3832} \approx 0.1495 \implies \mathbf{0.150} \text{ (3 d.p.)}$$

(c) The Newton-Raphson iteration requires $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. Differentiating $f(x)$:

$$f'(x) = \frac{3}{4}x^{-1/4} + x^{-1/2} = \frac{3}{4x^{1/4}} + \frac{1}{\sqrt{x}}$$

At $x_0 = 0$, both terms involve division by zero, so $f'(0)$ is undefined ($f'(x) \to \infty$ as $x \to 0^+$). Hence the tangent is vertical and Newton-Raphson cannot be computed.

(d) For all $x > 0$, $x^{1/4} > 0$ and $\sqrt{x} > 0$, hence $f'(x) = \frac{3}{4x^{1/4}} + \frac{1}{\sqrt{x}} > 0$. Since $f'(x) > 0$ everywhere on $(0, \infty)$, $f(x)$ is strictly monotonically increasing. A strictly increasing function can cross the x-axis at most once. Combined with part (a), α is the unique root on $[0, \infty)$.

Question 9. [Coordinate Geometry]

(a) $y^2 = 4ax$. Differentiate implicitly with respect to x:

$$2y\frac{dy}{dx} = 4a \implies \frac{dy}{dx} = \frac{2a}{y}$$

At point $P(ap^2, 2ap)$, the gradient is:

$$m = \frac{2a}{2ap} = \frac{1}{p}$$

Equation of the tangent at P:

$$y - 2ap = \frac{1}{p}(x - ap^2) \implies py - 2ap^2 = x - ap^2 \implies py = x + ap^2 \quad \text{as required.}$$

(b) Substitute $x = py - ap^2$ into the rectangular hyperbola $xy = c^2$:

$$(py - ap^2)y = c^2 \implies py^2 - ap^2y - c^2 = 0$$

Let the two intersection points be $A(x_1, y_1)$ and $B(x_2, y_2)$. The values y_1, y_2 are roots of the quadratic equation above. By Vieta's formulas:

$$y_1 + y_2 = \frac{ap^2}{p} = ap \implies y_M = \frac{y_1 + y_2}{2} = \frac{ap}{2}$$

Since $M(x_M, y_M)$ lies on the tangent line $x = py - ap^2$:

$$x_M = py_M - ap^2 = p\left(\frac{ap}{2}\right) - ap^2 = \frac{ap^2}{2} - ap^2 = -\frac{ap^2}{2}$$

Midpoint coordinates: $M\left(-\frac{ap^2}{2}, \frac{ap}{2}\right)$.

(c) From $y = \frac{ap}{2}$, we have $p = \frac{2y}{a}$. Substitute into the x-coordinate:

$$x = -\frac{a}{2}\left(\frac{2y}{a}\right)^2 = -\frac{a}{2}\left(\frac{4y^2}{a^2}\right) = -\frac{2y^2}{a} \implies ay = -2y^2 \implies 2y^2 + ax = 0$$

Notice that $y_M = \frac{ap}{2}$ and $x_M = -\frac{ap^2}{2}$ give $x_M + y_M = \frac{ap(1-p)}{2}$. If $a = c$, the condition for real intersections requires discriminant $\Delta = a^2p^4 + 4pc^2 > 0$. The locus of M is the parabola $y^2 = -\frac{a}{2}x$.