Further Pure Mathematics 2 (FP2)

Enhanced Practice Examination

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Time Allowed: 1 hour 30 minutes | **Total Marks:** 75 marks

Answer all questions. Show all necessary working; full marks may not be awarded without working.

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1. [First Order Differential Equations] [8 marks]
Consider the first order differential equation:

$$\frac{dy}{dx} + y\tan x = \sin x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

(a) Find the general solution of the differential equation, giving y explicitly in terms of x and an arbitrary constant c. [5]

(b) Find the particular solution satisfying the initial condition $y(0) = 0$. [3]

2. [Second Order Linear Differential Equations] [11 marks]
A damped oscillator is governed by the differential equation:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = \sin x$$

(a) Find the complementary function for the homogeneous equation. [3]

(b) By using a trial particular integral of the form $y_p = a\cos x + b\sin x$, determine the values of constants a and b. [4]

(c) State the general solution, and find the particular solution satisfying $y(0) = 0$ and $y'(0) = 0$. [4]

3. [Algebraic Inequalities] [8 marks]
Find the set of real values of x for which:

$$1 - x - x^2 > \frac{2}{x}$$

Show clear algebraic working, justifying all critical values and intervals. [8]

4. [Taylor and Maclaurin Series] [10 marks]

(a) A function $y = f(x)$ satisfies the non-linear differential equation:

$$\frac{d^2y}{dx^2} + y^2\frac{dy}{dx} = x$$

with initial conditions $y(1) = 0$ and $y'(1) = 1$. Show that $y''(1) = 1$ and $y'''(1) = 1$, and hence determine the Taylor series expansion of y in ascending powers of $(x-1)$ up to and including the term in $(x-1)^3$. [6]

(b) Given that a differentiable function $f(x)$ has Maclaurin series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$, prove that the Maclaurin series for $g(x) = f(nx)$ where n is a real constant is given by:

$$f(nx) = f(0) + nf'(0)x + \frac{n^2 f''(0)}{2!}x^2 + \cdots + \frac{n^k f^{(k)}(0)}{k!}x^k + \ldots [4]$$

5. [Hyperbolic Functions and Integration] [10 marks]

(a) Using the definition $\tanh x = \frac{\sinh x}{\cosh x}$ and standard hyperbolic identities, prove that:

$$\operatorname{sech}^2(2x) = \left(\frac{\tanh^2 x - 1}{\tanh^2 x + 1}\right)^2 \ [5]$$

(b) Evaluate the integral:

$$\int \frac{1}{\sqrt{8 - 2x - x^2}}\, dx$$

giving your answer in terms of an inverse trigonometric function. [5]

6. [Complex Numbers and Roots of Unity] [10 marks]

(a) Express $z = 3 - i$ in polar/exponential form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [3]

(b) Find the three cube roots of $3 - i$ in exact exponential form $Re^{i\phi}$. [3]

(c) Let $\omega = e^{i2\pi/n}$. Prove that $\sum_{k=0}^{n-1} \omega^k = 0$ for all integers $n \geq 2$, and interpret this result geometrically in the complex plane. [4]

7. [Polar Coordinates] [9 marks]
A curve has polar equation $r = e^{\theta} \sin\theta$ for $0 \leq \theta \leq \pi$.

(a) Find the exact polar coordinates (r, θ) of the non-zero points on the curve at which the tangent is parallel to the initial line $\theta = 0$. [6]

(b) Find the exact area of the region enclosed by the curve between $\theta = 0$ and $\theta = \frac{\pi}{2}$. [3]

8. [Summation of Series by Differences] [9 marks]

(a) Express $\frac{2}{(r-3)(r-1)}$ in partial fractions. [3]

(b) Hence evaluate the finite summation:

$$\sum_{r=5}^{25} \frac{2}{(r-3)(r-1)}$$

giving your answer as an exact single fraction in its lowest terms. [4]

(c) Find $\sum_{r=5}^{\infty} \frac{2}{(r-3)(r-1)}$. [2]

END OF EXAMINATION