Further Pure Mathematics 2 (FP2)

Complete Model Solutions & Commentary

Jon Baldie BSc (Hons), MSc

Module: Further Pure Mathematics 2 (FP2) | **Max Mark:** 75 marks

Question 1. [First Order Differential Equations]

(a) The differential equation is in standard linear form $\frac{dy}{dx} + P(x)y = Q(x)$:

$$P(x) = \tan x, \quad Q(x) = \sin x$$

The integrating factor $I(x)$ is:

$$I(x) = \exp\left(\int \tan x\, dx\right) = \exp(\ln|\sec x|) = \sec x$$

Multiply the entire differential equation by $\sec x$:

$$\sec x \frac{dy}{dx} + y \tan x \sec x = \sin x \sec x$$

The left hand side recognizes the product rule derivative $\frac{d}{dx}[y \sec x]$, and the right hand side is $\sin x \cdot \frac{1}{\cos x} = \tan x$:

$$\frac{d}{dx}(y \sec x) = \tan x$$

Integrate both sides with respect to x:

$$y \sec x = \int \tan x\, dx = \ln|\sec x| + c$$

Multiply both sides by $\cos x$ (noting that the constant of integration c must also be multiplied by $\cos x$):

$$y = \cos x(\ln|\sec x| + c) = \cos x \ln|\sec x| + c\cos x$$

General Solution: $y = \cos x \ln(\sec x) + c\cos x$ (since $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $\sec x > 0$).

(b) Applying the initial condition $y(0) = 0$:

$$0 = \cos(0)\ln(\sec 0) + c\cos(0) = 1 \cdot \ln(1) + c(1) = 0 + c \implies c = 0$$

Particular Solution: $y = \cos x \ln(\sec x)$.

Question 2. [Second Order Linear Differential Equations]

(a) Homogeneous equation: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$. Auxiliary equation:

$$\lambda^2 - 4\lambda + 4 = 0 \implies (\lambda - 2)^2 = 0 \implies \lambda = 2 \text{ (repeated root)}$$

The complementary function is:

$$y_{cf} = (A + Bx)e^{2x}$$

where A and B are arbitrary real constants.

(b) Trial particular integral: $y_p = a\cos x + b\sin x$. Differentiating:

$$y_p' = -a\sin x + b\cos x, \quad y_p'' = -a\cos x - b\sin x$$

Substitute into the left-hand side:

$$\begin{aligned} y_p'' - 4y_p' + 4y_p &= (-a\cos x - b\sin x) - 4(-a\sin x + b\cos x) + 4(a\cos x + b\sin x) \\ &= (-a - 4b + 4a)\cos x + (-b + 4a + 4b)\sin x \\ &= (3a - 4b)\cos x + (4a + 3b)\sin x \end{aligned}$$

Equating to the right-hand side, $\sin x$:

$$\begin{cases} 3a - 4b = 0 \implies a = \frac{4}{3}b \\ 4a + 3b = 1 \implies 4\left(\frac{4}{3}b\right) + 3b = \frac{25}{3}b = 1 \implies b = \frac{3}{25} \end{cases}$$

Hence $a = \frac{4}{3}\left(\frac{3}{25}\right) = \frac{4}{25}$. **Particular integral:** $y_p = \frac{4}{25}\cos x + \frac{3}{25}\sin x$.

(c) General solution:

$$y = (A + Bx)e^{2x} + \frac{4}{25}\cos x + \frac{3}{25}\sin x$$

From $y(0) = 0$:

$$(A + 0)e^0 + \frac{4}{25}(1) + 0 = 0 \implies A = -\frac{4}{25}$$

Differentiate using the product rule:

$$y' = Be^{2x} + 2(A + Bx)e^{2x} - \frac{4}{25}\sin x + \frac{3}{25}\cos x$$

Apply $y'(0) = 0$:

$$B(1) + 2A(1) - 0 + \frac{3}{25}(1) = 0 \implies B + 2\left(-\frac{4}{25}\right) + \frac{3}{25} = 0 \implies B - \frac{5}{25} = 0 \implies B = \frac{1}{5}$$

Particular solution: $y = \left(\frac{1}{5}x - \frac{4}{25}\right)e^{2x} + \frac{4}{25}\cos x + \frac{3}{25}\sin x$.

Question 3. [Algebraic Inequalities]

We wish to solve:

$$1 - x - x^2 > \frac{2}{x}$$

Bring all terms to one side over a common denominator:

$$\frac{2}{x} + x^2 + x - 1 < 0 \implies \frac{x^3 + x^2 - x + 2}{x} < 0$$

To factorise the cubic numerator $N(x) = x^3 + x^2 - x + 2$, test integer factors of 2:

$$N(-2) = (-2)^3 + (-2)^2 - (-2) + 2 = -8 + 4 + 2 + 2 = 0$$

Hence $(x + 2)$ is a factor. Performing polynomial division:

$$x^3 + x^2 - x + 2 = (x + 2)(x^2 - x + 1)$$

For the quadratic factor $x^2 - x + 1$, the discriminant is:

$$\Delta = (-1)^2 - 4(1)(1) = -3 < 0$$

Since the leading coefficient is positive and $\Delta < 0$, $x^2 - x + 1 > 0$ for all real x.

Therefore, the sign of the rational function is determined entirely by:

$$\frac{x + 2}{x} < 0$$

Critical values: $x = -2$ (root) and $x = 0$ (vertical asymptote).

Interval	$x < -2$	$-2 < x < 0$	$x > 0$
$x + 2$	$-$	$+$	$+$
x	$-$	$-$	$+$
$\frac{x+2}{x}$	$+$	$-$	$+$

We require the expression to be strictly negative (< 0).

Final Answer: The solution set is $-2 < x < 0$, or in interval notation $x \in (-2, 0)$.

Question 4. [Taylor and Maclaurin Series]

(a) We are given $y'' + y^2y' = x$ with $y(1) = 0$ and $y'(1) = 1$. At $x = 1$:

$$y''(1) + (0)^2(1) = 1 \implies y''(1) = 1$$

Differentiate the differential equation implicitly with respect to x, applying the product rule and chain rule to y^2y':

$$\frac{d}{dx}\left[y'' + y^2y'\right] = \frac{d}{dx}[x] \implies y''' + 2y(y')^2 + y^2y'' = 1$$

Evaluate at $x = 1$ using $y(1) = 0, y'(1) = 1, y''(1) = 1$:

$$y'''(1) + 2(0)(1)^2 + (0)^2(1) = 1 \implies y'''(1) = 1$$

The Taylor series of y about $x = 1$ is:

$$\begin{aligned} y(x) &= y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \frac{y'''(1)}{3!}(x-1)^3 + \dots \\ &= 0 + 1(x-1) + \frac{1}{2!}(x-1)^2 + \frac{1}{3!}(x-1)^3 + \dots \end{aligned}$$

Final Answer: $y \approx (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3$.

(b) Let $g(x) = f(nx)$. By the chain rule:

$$g'(x) = nf'(nx), \quad g''(x) = n^2 f''(nx), \quad \dots, \quad g^{(k)}(x) = n^k f^{(k)}(nx)$$

Evaluating each derivative at $x = 0$:

$$g(0) = f(0), \quad g'(0) = nf'(0), \quad g''(0) = n^2 f''(0), \quad \dots, \quad g^{(k)}(0) = n^k f^{(k)}(0)$$

Substituting into the Maclaurin expansion $g(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k$:

$$f(nx) = f(0) + nf'(0)x + \frac{n^2 f''(0)}{2!}x^2 + \cdots + \frac{n^k f^{(k)}(0)}{k!}x^k + \dots \blacksquare$$

Question 5. [Hyperbolic Functions and Integration]

(a) From $\cosh^2(2x) - \sinh^2(2x) = 1$, dividing by $\cosh^2(2x)$ gives $\operatorname{sech}^2(2x) = 1 - \tanh^2(2x)$. Using the hyperbolic double-angle identity:

$$\tanh(2x) = \frac{2\tanh x}{1+\tanh^2 x}$$

Substitute into the $\operatorname{sech}^2(2x)$ expression:

$$\begin{aligned}\operatorname{sech}^2(2x) = 1 - \left(\frac{2\tanh x}{1+\tanh^2 x}\right)^2 &= \frac{(1+\tanh^2 x)^2 - 4\tanh^2 x}{(1+\tanh^2 x)^2} \\ &= \frac{1+2\tanh^2 x + \tanh^4 x - 4\tanh^2 x}{(1+\tanh^2 x)^2} \\ &= \frac{\tanh^4 x - 2\tanh^2 x + 1}{(1+\tanh^2 x)^2} = \frac{(\tanh^2 x - 1)^2}{(\tanh^2 x + 1)^2} = \left(\frac{\tanh^2 x - 1}{\tanh^2 x + 1}\right)^2 \quad \blacksquare\end{aligned}$$

(b) Completing the square in the quadratic denominator:

$$8 - 2x - x^2 = 8 - (x^2+2x) = 8 - \left[(x+1)^2 - 1\right] = 9 - (x+1)^2 = 3^2 - (x+1)^2$$

Let $u = x+1$, so $du = dx$. The integral becomes:

$$\int \frac{1}{\sqrt{3^2 - u^2}}\,du = \arcsin\left(\frac{u}{3}\right) + C = \arcsin\left(\frac{x+1}{3}\right) + C$$

Final Answer: $\arcsin\left(\frac{x+1}{3}\right) + C$.

Question 6. [Complex Numbers and Roots of Unity]

(a) For $z = 3 - i$: Modulus: $r = |z| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$. Argument: $\theta = \arctan\left(-\frac{1}{3}\right) = -\arctan\left(\frac{1}{3}\right) \approx -0.322$ rad.

$$z = \sqrt{10}\exp\left(-i\arctan\frac{1}{3}\right)$$

(b) By de Moivre's theorem, the cube roots $w_k = z^{1/3}$ for $k \in \{0,1,2\}$ have modulus:

$$R = (\sqrt{10})^{1/3} = 10^{1/6}$$

and arguments:

$$\phi_k = \frac{\theta + 2k\pi}{3} = \frac{-\arctan(1/3) + 2k\pi}{3}$$

Cube roots: $w_k = 10^{1/6}\exp\left[i\left(\frac{-\arctan(1/3)+2k\pi}{3}\right)\right]$ for $k = 0, 1, 2$.

(c) The sum $S = \sum_{k=0}^{n-1}\omega^k = 1 + \omega + \omega^2 + \cdots + \omega^{n-1}$ is a geometric series with common ratio $\omega = e^{i2\pi/n} \neq 1$:

$$S = \frac{1-\omega^n}{1-\omega}$$

Since $\omega^n = (e^{i2\pi/n})^n = e^{i2\pi} = 1$:

$$S = \frac{1-1}{1-\omega} = 0 \quad \blacksquare$$

Geometric interpretation: The n roots of unity represent the vertices of a regular n-gon centered at the origin of the Argand plane. Their vector sum represents the centroid (center of mass) of the system, which lies identically at the origin due to rotational symmetry.

Question 7. [Polar Coordinates]

(a) In polar coordinates, $y = r\sin\theta$. With $r = e^\theta \sin\theta$:

$$y = e^\theta \sin^2\theta$$

Tangents parallel to the initial line have $\frac{dy}{d\theta} = 0$:

$$\frac{dy}{d\theta} = e^\theta \sin^2\theta + 2e^\theta \sin\theta\cos\theta = e^\theta \sin\theta(\sin\theta + 2\cos\theta) = 0$$

Since $e^\theta \neq 0$:

- $\sin\theta = 0 \implies \theta = 0$ or $\theta = \pi$. At both values, $r = 0$ (the pole).
- $\sin\theta + 2\cos\theta = 0 \implies \tan\theta = -2$.

In the domain $0 \leq \theta \leq \pi$, $\tan\theta = -2$ occurs in the second quadrant:

$$\theta_1 = \pi - \arctan(2) \approx 2.034 \text{ rad}$$

At this angle, $\sin\theta_1 = \frac{2}{\sqrt{5}}$. The radial coordinate is:

$$r_1 = e^{\theta_1}\sin\theta_1 = \frac{2}{\sqrt{5}}e^{\pi - \arctan(2)}$$

Point: $\left(\frac{2}{\sqrt{5}}e^{\pi-\arctan(2)},\, \pi - \arctan(2)\right) \approx (6.84,\, 2.034 \text{ rad})$.

(b) The area of a sector in polar coordinates is:

$$A = \frac{1}{2}\int_0^{\pi/2} r^2\, d\theta = \frac{1}{2}\int_0^{\pi/2} e^{2\theta}\sin^2\theta\, d\theta$$

Using $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$:

$$A = \frac{1}{4}\int_0^{\pi/2} e^{2\theta}(1 - \cos 2\theta)\, d\theta = \frac{1}{4}\left[\int_0^{\pi/2} e^{2\theta}\, d\theta - \int_0^{\pi/2} e^{2\theta}\cos(2\theta)\, d\theta\right]$$

First integral: $\int_0^{\pi/2} e^{2\theta}\, d\theta = \left[\frac{e^{2\theta}}{2}\right]_0^{\pi/2} = \frac{e^\pi - 1}{2}$. Second integral: $\int e^{2\theta}\cos(2\theta)\, d\theta = \frac{e^{2\theta}}{8}(2\cos 2\theta + 2\sin 2\theta) = \frac{e^{2\theta}}{4}(\cos 2\theta + \sin 2\theta)$. Evaluated from 0 to $\pi/2$:

$$\left[\frac{e^{2\theta}}{4}(\cos 2\theta + \sin 2\theta)\right]_0^{\pi/2} = \frac{e^\pi}{4}(-1+0) - \frac{1}{4}(1+0) = -\frac{e^\pi + 1}{4}$$

Combining:

$$A = \frac{1}{4}\left[\frac{e^\pi - 1}{2} - \left(-\frac{e^\pi + 1}{4}\right)\right] = \frac{1}{4}\left[\frac{2e^\pi - 2 + e^\pi + 1}{4}\right] = \frac{3e^\pi - 1}{16}$$

Final Answer: $A = \frac{3e^\pi - 1}{16} \approx 4.27$.

Question 8. [Summation of Series by Differences]

(a) $\frac{2}{(r-3)(r-1)} = \frac{A}{r-3} + \frac{B}{r-1}$. $2 = A(r-1) + B(r-3)$. Setting $r = 3 \implies 2 = 2A \implies A = 1$. Setting $r = 1 \implies 2 = -2B \implies B = -1$.

$$\frac{2}{(r-3)(r-1)} = \frac{1}{r-3} - \frac{1}{r-1}$$

(b) Expand the telescoping sum:

$$\sum_{r=5}^{25} \left(\frac{1}{r-3} - \frac{1}{r-1} \right) = \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \dots$$
$$+ \left(\frac{1}{21} - \frac{1}{23} \right) + \left(\frac{1}{22} - \frac{1}{24} \right)$$

All intermediate terms cancel, leaving the first two positive terms and last two negative terms:

$$S = \frac{1}{2} + \frac{1}{3} - \frac{1}{23} - \frac{1}{24} = \left(\frac{1}{2} - \frac{1}{24} \right) + \frac{1}{3} - \frac{1}{23} = \frac{11}{24} + \frac{8}{24} - \frac{1}{23} = \frac{19}{24} - \frac{1}{23}$$

$$S = \frac{19 \times 23 - 24}{24 \times 23} = \frac{437 - 24}{552} = \frac{413}{552}$$

(c) As $N \to \infty$, the negative tail terms $\frac{1}{N-1}$ and $\frac{1}{N}$ tend to zero:

$$\sum_{r=5}^{\infty} \frac{2}{(r-3)(r-1)} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$