Further Pure Mathematics 3 (FP3 MEI)

Enhanced Practice Examination

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Time Allowed: 1 hour 30 minutes | **Total Marks:** 75 marks

Answer all questions. Show all necessary working; full marks may not be awarded without working.

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1. [3D Vectors: Cross Product and Planes] [9 marks]
The points $A(1, 0, 2)$, $B(2, -1, 3)$, and $C(0, 3, 1)$ lie in three-dimensional space $\mathbb{R}^3$.

(a) Find the vectors $\vec{AB}$ and $\vec{AC}$. [2]

(b) Hence determine a vector perpendicular to the plane Π containing A, B, and C. [3]

(c) Find the Cartesian equation of the plane Π in the form $ax + by + cz = d$. [2]

(d) Find the exact perpendicular distance from the origin $(0, 0, 0)$ to the plane Π. [2]

2. [Vectors: Shortest Distance Between Skew Lines] [10 marks]
Two lines L_1 and L_2 have vector equations:

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad L_2 : \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

(a) Show that L_1 and L_2 are not parallel and do not intersect, and hence deduce that they are skew lines. [5]

(b) Find the shortest distance between L_1 and L_2. [5]

3. [Multivariable Calculus: Stationary Points and Classification] [12 marks]
A surface is defined by the function:

$$z = f(x, y) = x^3 + y^3 - 3xy + 4$$

(a) Find the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, and $\frac{\partial^2 f}{\partial x \partial y}$. [4]

(b) Find the coordinates of the two stationary points of $f(x, y)$. [3]

(c) By evaluating the discriminant $\Delta = f_{xx} f_{yy} - (f_{xy})^2$ at each stationary point, classify each point as a local minimum, local maximum, or saddle point. [5]

4. [Multivariable Calculus: Tangent Planes] [8 marks]
A surface S has equation $x^2 + 2y^2 + 3z^2 = 36$.

(a) Find a normal vector to the surface S at the point $P(1, 2, 3)$. [4]

(b) Hence write down the Cartesian equation of the tangent plane to S at P. [2]

(c) Find the coordinates of the point where this tangent plane intersects the z-axis. [2]

5. [Matrix Algebra: Diagonalisation] [12 marks]
Let $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$.

(a) Find the eigenvalues λ_1 and λ_2 of $\mathbf{A}$. [4]

(b) For each eigenvalue, find a corresponding eigenvector. [4]

(c) Write down an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$. [2]

(d) Hence calculate $\mathbf{A}^n$ for any positive integer $n \geq 1$. [2]

6. [Invariant Lines] [10 marks]
A linear transformation in $\mathbb{R}^2$ is represented by the matrix:

$$\mathbf{M} = \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix}$$

(a) Show that the eigenvalues of $\mathbf{M}$ are $\lambda = 2$ and $\lambda = 3$. [3]

(b) Explain clearly the distinction between an *invariant line* and a *line of invariant points*. [2]

(c) Find Cartesian equations for the two invariant lines through the origin under $\mathbf{M}$. [5]

7. [Markov Chains: Transition Matrices and Steady State] [14 marks]
A subscription software product tracks its customers across three states each month: Active (A), Paused (P), and Churned (C). Churned is an absorbing state. A simplified 2-state active model between Active (A) and Inactive (I) has monthly transition matrix:

$$\mathbf{T} = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$

where row 1 represents transition probabilities from State A to (A, I), and row 2 represents transition probabilities from State I to (A, I).

(a) At month 0, 100% of users are in State A, so the initial state probability vector is $\mathbf{p}_0 = \begin{pmatrix} 1 & 0 \end{pmatrix}$. Calculate the state probability distribution $\mathbf{p}_2$ after two months. [4]

(b) Let $\pi = \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix}$ be the steady-state probability vector satisfying $\pi\mathbf{T} = \pi$ and $\pi_1 + \pi_2 = 1$. Determine the exact values of π_1 and π_2. [6]

(c) Interpret the long-term equilibrium proportion of active users for this software service. [4]

END OF EXAMINATION