Further Pure Mathematics 3 (FP3 MEI)

Complete Model Solutions & Commentary

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Module: Further Pure Mathematics 3 (FP3 MEI) | **Max Mark:** 75 marks

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Question 1. [3D Vectors: Cross Product and Planes]

(a) With $A(1,0,2)$, $B(2,-1,3)$, and $C(0,3,1)$:

$$\vec{AB} = \begin{pmatrix} 2-1 \\ -1-0 \\ 3-2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \vec{AC} = \begin{pmatrix} 0-1 \\ 3-0 \\ 1-2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$$

(b) The normal vector $\mathbf{n}$ is obtained via the cross product $\vec{AB} \times \vec{AC}$:

$$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} (-1)(-1)-(1)(3) \\ (1)(-1)-(1)(-1) \\ (1)(3)-(-1)(-1) \end{pmatrix} = \begin{pmatrix} 1-3 \\ -1+1 \\ 3-1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$$

Scaling by $-\frac{1}{2}$ for simplicity, we can use $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

(c) The plane equation is $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$:

$$d = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = (1)(1) + (0)(0) + (2)(-1) = 1 - 2 = -1$$

Cartesian equation: $x - z = -1 \implies x - z + 1 = 0$ (or $-2x + 2z = 2$).

(d) The perpendicular distance from point $(x_0, y_0, z_0) = (0,0,0)$ to plane $ax+by+cz = d$ is:

$$D = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2+b^2+c^2}} = \frac{|0-0-(-1)|}{\sqrt{1^2+0^2+(-1)^2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Question 2. [Vectors: Shortest Distance Between Skew Lines]

(a) Direction vectors: $\mathbf{d}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{d}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$. Since $\mathbf{d}_1 \neq k\mathbf{d}_2$ for any scalar k, the lines are not parallel. Equating coordinates for intersection:

$$\begin{pmatrix} 1+2\lambda \\ 2-\lambda \\ -1+\lambda \end{pmatrix} = \begin{pmatrix} 3+\mu \\ -1+\mu \\ 2-2\mu \end{pmatrix}$$

From the second equation: $\mu = 3 - \lambda$. Substitute into the first equation:

$$1 + 2\lambda = 3 + (3 - \lambda) = 6 - \lambda \implies 3\lambda = 5 \implies \lambda = \frac{5}{3}, \quad \mu = \frac{4}{3}$$

Check the third equation:

$$\text{LHS} = -1 + \lambda = -1 + \frac{5}{3} = \frac{2}{3}, \quad \text{RHS} = 2 - 2\mu = 2 - 2\left(\frac{4}{3}\right) = 2 - \frac{8}{3} = -\frac{2}{3}$$

Since $\frac{2}{3} \neq -\frac{2}{3}$, the system is inconsistent and the lines do not intersect. Lines that are neither parallel nor intersecting are skew lines. ■

(b) Common perpendicular vector $\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2$:

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} (-1)(-2) - (1)(1) \\ (1)(1) - (2)(-2) \\ (2)(1) - (-1)(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$

Vector connecting point $\mathbf{a}_1 = (1, 2, -1)$ on L_1 to point $\mathbf{a}_2 = (3, -1, 2)$ on L_2:

$$\mathbf{a}_2 - \mathbf{a}_1 = \begin{pmatrix} 3 - 1 \\ -1 - 2 \\ 2 - (-1) \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$$

The shortest distance is the scalar projection:

$$d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|(2)(1) + (-3)(5) + (3)(3)|}{\sqrt{1^2 + 5^2 + 3^2}} = \frac{|2 - 15 + 9|}{\sqrt{1 + 25 + 9}} = \frac{|-4|}{\sqrt{35}} = \frac{4}{\sqrt{35}} = \frac{4\sqrt{35}}{35}$$

Question 3. [Multivariable Calculus: Stationary Points and Classification]

(a) Given $f(x, y) = x^3 + y^3 - 3xy + 4$:

$$f_x = \frac{\partial f}{\partial x} = 3x^2 - 3y, \quad f_y = \frac{\partial f}{\partial y} = 3y^2 - 3x$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 6x, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = 6y, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = -3$$

(b) Setting partial derivatives to zero:

$$3x^2 - 3y = 0 \implies y = x^2 \quad \text{and} \quad 3y^2 - 3x = 0 \implies x = y^2$$

Substitute $y = x^2$ into $x = y^2$:

$$x = (x^2)^2 = x^4 \implies x^4 - x = 0 \implies x(x^3 - 1) = 0 \implies x(x-1)(x^2 + x + 1) = 0$$

Real solutions: $x = 0 \implies y = 0$; and $x = 1 \implies y = 1$. The stationary points are $(0, 0)$ and $(1, 1)$.

(c) Discriminant (Hessian determinant): $\Delta(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = (6x)(6y) - (-3)^2 = 36xy - 9$.

- **At** $(0, 0)$**:** $\Delta(0, 0) = 0 - 9 = -9 < 0$. Since $\Delta < 0$, $(0, 0)$ is a **saddle point**.
- **At** $(1, 1)$**:** $\Delta(1, 1) = 36(1)(1) - 9 = 27 > 0$, and $f_{xx}(1, 1) = 6(1) = 6 > 0$. Since $\Delta > 0$ and $f_{xx} > 0$, $(1, 1)$ is a **local minimum**. Value at minimum: $f(1, 1) = 1^3 + 1^3 - 3(1)(1) + 4 = 3$.

Question 4. [Multivariable Calculus: Tangent Planes]

(a) Define $F(x, y, z) = x^2 + 2y^2 + 3z^2 - 36 = 0$. The gradient vector ∇F is normal to the level surface:

$$\nabla F = \begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x \\ 4y \\ 6z \end{pmatrix}$$

At $P(1, 2, 3)$:

$$\mathbf{n} = \begin{pmatrix} 2(1) \\ 4(2) \\ 6(3) \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 18 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}$$

(b) The tangent plane has normal $\begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}$ and passes through $(1, 2, 3)$:

$$1(x-1) + 4(y-2) + 9(z-3) = 0 \implies x + 4y + 9z = 1 + 8 + 27 = 36$$

Equation of tangent plane: $x + 4y + 9z = 36$.

(c) On the z-axis, $x = 0$ and $y = 0$:

$$0 + 0 + 9z = 36 \implies z = 4$$

Intersection point: $(0, 0, 4)$.

Question 5. [Matrix Algebra: Diagonalisation]

(a) Characteristic equation $\det(\mathbf{A}-\lambda\mathbf{I})=0$:

$$\det\begin{pmatrix}4-\lambda & 2\\ 1 & 3-\lambda\end{pmatrix}=(4-\lambda)(3-\lambda)-2=\lambda^2-7\lambda+10=(\lambda-2)(\lambda-5)=0$$

Eigenvalues: $\lambda_1=2, \quad \lambda_2=5$.

(b)

- **For $\lambda_1=2$:** $(\mathbf{A}-2\mathbf{I})\mathbf{v}=\begin{pmatrix}2&2\\1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\implies x+y=0\implies \mathbf{v}_1=\begin{pmatrix}1\\-1\end{pmatrix}$.
- **For $\lambda_2=5$:** $(\mathbf{A}-5\mathbf{I})\mathbf{v}=\begin{pmatrix}-1&2\\1&-2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\implies -x+2y=0\implies$ $\mathbf{v}_2=\begin{pmatrix}2\\1\end{pmatrix}$.

(c) Matrix of eigenvectors $\mathbf{P}$ and diagonal matrix $\mathbf{D}$:

$$\mathbf{P}=\begin{pmatrix}1&2\\-1&1\end{pmatrix},\quad \mathbf{D}=\begin{pmatrix}2&0\\0&5\end{pmatrix}$$

(d) We have $\det(\mathbf{P})=(1)(1)-(2)(-1)=3$, so $\mathbf{P}^{-1}=\frac{1}{3}\begin{pmatrix}1&-2\\1&1\end{pmatrix}$.

$$\mathbf{A}^n=\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}=\begin{pmatrix}1&2\\-1&1\end{pmatrix}\begin{pmatrix}2^n&0\\0&5^n\end{pmatrix}\frac{1}{3}\begin{pmatrix}1&-2\\1&1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}2^n+2\cdot5^n & -2^{n+1}+2\cdot5^n\\ -2^n+5^n & 2^{n+1}+5^n\end{pmatrix}$$

Question 6. [Invariant Lines]

(a) $\det(\mathbf{M}-\lambda\mathbf{I})=(-1-\lambda)(6-\lambda)-(2)(-6)=\lambda^2-5\lambda-6+12=\lambda^2-5\lambda+6=(\lambda-2)(\lambda-3)=0$. The roots are $\lambda=2$ and $\lambda=3$. ■

(b)

- **Line of invariant points:** Every individual point on the line is mapped to itself: $\mathbf{Mx}=\mathbf{x}$. This corresponds to eigenvalue $\lambda=1$. Since neither eigenvalue is 1, there are no lines of invariant points.
- **Invariant line:** The line as a whole is mapped onto itself. Points on the line may move along the line: $\mathbf{Mx}=\lambda\mathbf{x}$. For lines through the origin, these lie along the eigenvector directions.

(c) For lines through the origin $y=mx$:

- **For $\lambda=2$:** $(\mathbf{M}-2\mathbf{I})\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-3&2\\-6&4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\implies -3x+2y=0\implies$ $y=\frac{3}{2}x$.
- **For $\lambda=3$:** $(\mathbf{M}-3\mathbf{I})\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-4&2\\-6&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\implies -4x+2y=0\implies$ $y=2x$.

The two invariant lines: $y=\frac{3}{2}x$ (or $3x-2y=0$) and $y=2x$ (or $2x-y=0$).

Question 7. [Markov Chains: Transition Matrices and Steady State]

(a) Initial vector $\mathbf{p}_0 = \begin{pmatrix} 1 & 0 \end{pmatrix}$. Month 1:

$$\mathbf{p}_1 = \mathbf{p}_0\mathbf{T} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.2 \end{pmatrix}$$

Month 2:

$$\mathbf{p}_2 = \mathbf{p}_1\mathbf{T} = \begin{pmatrix} 0.8 & 0.2 \end{pmatrix} \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix} = \begin{pmatrix} 0.8(0.8) + 0.2(0.3) & 0.8(0.2) + 0.2(0.7) \end{pmatrix} = \begin{pmatrix} 0.70 & 0.30 \end{pmatrix}$$

After two months, 70% of users remain Active and 30% are Inactive.

(b) The steady-state vector $\pi = \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix}$ satisfies:

$$\begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix}$$

Expanding the first column:

$$0.8\pi_1 + 0.3\pi_2 = \pi_1 \implies 0.3\pi_2 = 0.2\pi_1 \implies 2\pi_1 = 3\pi_2 \implies \pi_1 = \frac{3}{2}\pi_2$$

Since total probability must sum to 1:

$$\pi_1 + \pi_2 = 1 \implies \frac{3}{2}\pi_2 + \pi_2 = 1 \implies \frac{5}{2}\pi_2 = 1 \implies \pi_2 = \frac{2}{5} = 0.4$$

$$\pi_1 = 1 - 0.4 = \frac{3}{5} = 0.6$$

Steady-state vector: $\pi = \begin{pmatrix} 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \end{pmatrix}$.

(c) In the long run, regardless of the initial customer cohort breakdown, the subscriber base approaches a dynamic equilibrium in which **60% of users are Active** and **40% are Inactive**. The churn to inactivity ($0.2 \times 60\% = 12\%$ of total users) exactly balances reactivations ($0.3 \times 40\% = 12\%$ of total users).