Mechanics 1 (M1)

Complete Model Solutions & Commentary

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Module: Mechanics 1 (M1) | **Max Mark:** 75 marks | $g = 9.8$ m s^{-2}

Question 1. [Kinematics with Variable Acceleration]

(a) Velocity $v(t) = \int a(t)\,dt = \int (6t - 18)\,dt = 3t^2 - 18t + c$. Given initial velocity $v(0) = 24$ m s^{-1}:

$$24 = 3(0)^2 - 18(0) + c \implies c = 24$$

Velocity expression: $v(t) = 3t^2 - 18t + 24$ m s^{-1}.

(b) Instantaneously at rest when $v(t) = 0$:

$$3t^2 - 18t + 24 = 0 \implies 3(t^2 - 6t + 8) = 0 \implies 3(t-2)(t-4) = 0$$

Times at rest: $t = 2$ s and $t = 4$ s.

(c) Displacement $s(t) = \int v(t)\,dt = t^3 - 9t^2 + 24t + C_0$. With $s(0) = 0$, $s(t) = t^3 - 9t^2 + 24t$. Evaluate position at key timestamps:

- $s(0) = 0$ m
- $s(2) = (2)^3 - 9(2)^2 + 24(2) = 8 - 36 + 48 = 20$ m
- $s(4) = (4)^3 - 9(4)^2 + 24(4) = 64 - 144 + 96 = 16$ m
- $s(5) = (5)^3 - 9(5)^2 + 24(5) = 125 - 225 + 120 = 20$ m

Total distance travelled:

$$D = |s(2) - s(0)| + |s(4) - s(2)| + |s(5) - s(4)| = |20 - 0| + |16 - 20| + |20 - 16| = 20 + 4 + 4 = 28 \text{ m}$$

Final Answer: 28 m.

Question 2. [Connected Particles over a Smooth Pulley]

(a) Particle B (5 kg) accelerates downwards; particle A (3 kg) accelerates upwards with magnitude a. Equations of motion ($F = ma$):

$$\begin{aligned} \text{For } A: &\quad T - 3g = 3a \implies T = 3g + 3a \\ \text{For } B: &\quad 5g - T = 5a \end{aligned}$$

Adding the equations:

$$5g - 3g = (3+5)a \implies 2g = 8a \implies a = \frac{2}{8}g = \frac{g}{4} = \frac{9.8}{4} = 2.45 \text{ m s}^{-2}$$

Substitute into tension:

$$T = 3g + 3(0.25g) = 3.75g = 3.75(9.8) = 36.75 \text{ N}$$

Values: $a = 2.45 \text{ m s}^{-2}$, $\quad T = 36.75$ N.

(b) B starts from rest ($u = 0$), moves distance $s = 2$ m with acceleration $a = 2.45 \text{ m s}^{-2}$:

$$v^2 = u^2 + 2as = 0 + 2(2.45)(2) = 9.8 \implies v = \sqrt{9.8} \approx 3.13 \text{ m s}^{-1}$$

(c) When B hits the ground, A is at height $h_1 = 2 + 2 = 4$ m above the ground, moving upwards with speed $v = \sqrt{9.8} \text{ m s}^{-1}$. The string becomes slack; A moves freely under gravity with deceleration $g = 9.8 \text{ m s}^{-2}$:

$$v_{\text{top}}^2 = v^2 - 2gh_2 \implies 0 = 9.8 - 2(9.8)h_2 \implies 2(9.8)h_2 = 9.8 \implies h_2 = 0.5 \text{ m}$$

Total maximum height above ground:

$$H_{\max} = 4 \text{ m} + 0.5 \text{ m} = 4.5 \text{ m}$$

Question 3. [Dynamics on an Inclined Rough Plane]

(a) $\tan\theta = \frac{3}{4} \implies \sin\theta = \frac{3}{5} = 0.6, \quad \cos\theta = \frac{4}{5} = 0.8$. Normal reaction: $R = mg\cos\theta = 4(9.8)(0.8) = 31.36$ N. Maximum friction force: $F_{\max} = \mu R = 0.25(31.36) = 7.84$ N. Component of weight acting down the plane:

$$W_{\parallel} = mg\sin\theta = 4(9.8)(0.6) = 23.52 \text{ N}$$

Since $W_{\parallel} = 23.52 \text{ N} > F_{\max} = 7.84$ N, gravity overcomes friction and the package accelerates down the plane:

$$mg\sin\theta - F_{\max} = ma \implies 23.52 - 7.84 = 4a \implies 15.68 = 4a \implies a = 3.92 \text{ m s}^{-2}$$

(b) A horizontal force P is applied towards the plane. Resolving forces: Perpendicular to plane: $R = mg\cos\theta + P\sin\theta$. Parallel to plane (preventing downward slip, so friction F acts up the plane at maximum limiting value $F = \mu R$):

$$P\cos\theta + F = mg\sin\theta \implies P\cos\theta + \mu(mg\cos\theta + P\sin\theta) = mg\sin\theta$$

Collecting terms in P:

$$P(\cos\theta + \mu\sin\theta) = mg(\sin\theta - \mu\cos\theta)$$

Substitute numerical values:

$$\cos\theta + \mu\sin\theta = 0.8 + 0.25(0.6) = 0.8 + 0.15 = 0.95$$

$$\sin\theta - \mu\cos\theta = 0.6 - 0.25(0.8) = 0.6 - 0.20 = 0.40$$

$$P(0.95) = 4(9.8)(0.40) = 15.68 \implies P = \frac{15.68}{0.95} = \frac{1568}{95} \approx 16.51 \text{ N}$$

Minimum force: $P \approx 16.5$ N (or $\frac{1568}{95}$ N).

Question 4. [Statics: Coplanar Equilibrium of a Suspended Body]

(a) The ring R is smooth, so it can slide freely along the continuous string. There is no friction at the ring, which means the tension must be uniform throughout the single continuous string, so $T_{AR} = T_{BR} = T$.

(b) By horizontal equilibrium, $T\sin\theta_A = T\sin\theta_B \implies \theta_A = \theta_B = \theta$. Thus triangle ABR is isosceles with $AR = BR = \frac{14}{2} = 7$ m. The horizontal distance from A to the midpoint is $\frac{10}{2} = 5$ m. By Pythagoras' theorem, the vertical depth h is:

$$h = \sqrt{7^2 - 5^2} = \sqrt{49 - 25} = \sqrt{24} = 2\sqrt{6} \text{ m} \approx 4.90 \text{ m}$$

(c) Resolving forces vertically for the ring:

$$2T\cos\phi = mg = 12(9.8) = 117.6 \text{ N}$$

where ϕ is the angle with the vertical, so $\cos\phi = \frac{h}{7} = \frac{\sqrt{24}}{7}$.

$$2T\left(\frac{\sqrt{24}}{7}\right) = 117.6 \implies T = \frac{117.6 \times 7}{2\sqrt{24}} = \frac{823.2}{4\sqrt{6}} = \frac{205.8}{\sqrt{6}} \approx 84.0 \text{ N}$$

Tension: $T = \frac{1029\sqrt{6}}{30} = 34.3\sqrt{6}$ N ≈ 84.0 N.

Question 5. [Moments: Non-Uniform Rigid Rod]

(a) Length $AB = 6$ m, mass $M = 40$ kg, weight $W = 40g$. Supports at C (1 m from A) and D (5 m from A, so $CD = 4$ m). Let the center of mass G be at distance $\bar{x}$ from A. When mass $m = 20$ kg is placed at A, the rod is on the point of tilting about $C \implies R_D = 0$. Taking moments about C:

$$\sum M_C = 0 \implies (20g)(1) = (40g)(\bar{x}-1) \implies 20 = 40(\bar{x}-1) \implies \bar{x}-1 = 0.5 \implies \bar{x} = 1.5 \text{ m}$$

The center of mass is 1.5 m from A.

(b) Force F at B (6 m from A). The rod is on the point of tilting about D (5 m from A) $\implies R_C = 0$. Taking moments about D: The center of mass G is at 1.5 m from A, so its distance from D is $5 - 1.5 = 3.5$ m to the left. End B is at 6 m from A, so its distance from D is $6 - 5 = 1$ m to the right.

$$(40g)(3.5) = F(1) \implies F = 140g = 140(9.8) = 1372 \text{ N}$$

(c) Unloaded rod: $R_C + R_D = 40g = 392$ N. Taking moments about C:

$$(40g)(1.5 - 1) = R_D(4) \implies (40g)(0.5) = 4R_D \implies 20g = 4R_D \implies R_D = 5g = 49 \text{ N}$$

$$R_C = 40g - R_D = 35g = 35(9.8) = 343 \text{ N}$$

Reactions: $R_C = 343$ N, $R_D = 49$ N.

Question 6. [2D Vectors and Projectiles]

(a) $\mathbf{v}(t) = (4 - 3t)\mathbf{i} + (2t - 5)\mathbf{j}$. Acceleration: $\mathbf{a} = \frac{d\mathbf{v}}{dt} = -3\mathbf{i} + 2\mathbf{j}$ m s^{-2}. Resultant force: $\mathbf{F} = m\mathbf{a} = 0.5(-3\mathbf{i}+2\mathbf{j}) = -1.5\mathbf{i}+\mathbf{j}$ N. Magnitude: $|\mathbf{F}| = \sqrt{(-1.5)^2 + 1^2} = \sqrt{2.25 + 1} = \sqrt{3.25} \approx 1.80$ N. Direction: $\arctan\left(\frac{1}{-1.5}\right) \implies 180° - 33.7° = 146.3°$ to the positive x-axis.

(b) Direction $2\mathbf{i} - \mathbf{j}$ requires velocity components to be in ratio $\frac{v_y}{v_x} = \frac{-1}{2}$:

$$\frac{2t - 5}{4 - 3t} = -\frac{1}{2} \implies 2(2t - 5) = -(4 - 3t) \implies 4t - 10 = -4 + 3t \implies t = 6 \text{ s}$$

At $t = 6$: $v_x = 4 - 18 = -14$, $v_y = 12 - 5 = 7$, so $\mathbf{v} = -14\mathbf{i} + 7\mathbf{j} = -7(2\mathbf{i} - \mathbf{j})$ (opposite direction), or for strictly the same direction t requires positive scalar multiple. When $\mathbf{v}$ is parallel, $t = 6$ s.

(c) At $t = 2$ s:

$$\mathbf{v}(2) = (4 - 6)\mathbf{i} + (4 - 5)\mathbf{j} = -2\mathbf{i} - \mathbf{j} \text{ m s}^{-1}$$

Speed squared: $v^2 = (-2)^2 + (-1)^2 = 4 + 1 = 5$ m^2 s^{-2}. Kinetic energy:

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}(0.5)(5) = 1.25 \text{ J}$$

Question 7. [Momentum and Direct Collisions]

(a) Masses: $m_P = m, \quad m_Q = 2m$. Initial velocities: $u_P = 4u, \quad u_Q = u$. Let velocities immediately after impact be v_P and v_Q. Conservation of linear momentum:

$$m(4u) + 2m(u) = mv_P + 2mv_Q \implies 6mu = m(v_P + 2v_Q) \implies v_P + 2v_Q = 6u \quad \text{— (1)}$$

Newton's law of restitution:

$$v_Q - v_P = e(u_P - u_Q) = \frac{1}{2}(4u - u) = \frac{3}{2}u \implies -v_P + v_Q = \frac{3}{2}u \quad \text{— (2)}$$

Adding (1) and (2):

$$3v_Q = 6u + \frac{3}{2}u = \frac{15}{2}u \implies v_Q = \frac{5}{2}u$$

Wait, let's check: If $v_Q = \frac{5}{2}u$, then $v_P = v_Q - \frac{3}{2}u = \frac{5}{2}u - \frac{3}{2}u = u$. Let's check conservation of momentum: $1(u) + 2(2.5u) = u + 5u = 6u$. Restitution: $v_Q - v_P = 2.5u - u = 1.5u = 0.5(4u - u) = 1.5u$. Therefore, $v_P = u$ and $v_Q = \frac{5}{2}u$. **Note on problem statement:** Taking $v_P = u$ and $v_Q = \frac{5}{2}u$:

(b) The impulse I received by sphere Q:

$$I = \Delta p_Q = m_Q(v_Q - u_Q) = 2m\left(\frac{5}{2}u - u\right) = 2m\left(\frac{3}{2}u\right) = 3mu \text{ N s}$$

(c) Initial kinetic energy:

$$E_{k,\text{init}} = \frac{1}{2}m(4u)^2 + \frac{1}{2}(2m)(u)^2 = \frac{1}{2}m(16u^2) + mu^2 = 8mu^2 + mu^2 = 9mu^2$$

Final kinetic energy:

$$E_{k,\text{final}} = \frac{1}{2}m(u)^2 + \frac{1}{2}(2m)\left(\frac{5}{2}u\right)^2 = \frac{1}{2}mu^2 + m\left(\frac{25}{4}u^2\right) = \left(\frac{1}{2} + \frac{25}{4}\right)mu^2 = \frac{27}{4}mu^2$$

Kinetic energy lost:

$$\Delta E_k = 9mu^2 - \frac{27}{4}mu^2 = \frac{36 - 27}{4}mu^2 = \frac{9}{4}mu^2$$

Fraction of initial kinetic energy lost:

$$\frac{\Delta E_k}{E_{k,\text{init}}} = \frac{\frac{9}{4}mu^2}{9mu^2} = \frac{1}{4} = 25\%$$

Fraction lost: $\frac{1}{4}$ (or 25%).